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A new nonlinear functional for general scalar conservation laws

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ABSTRACT

The approach based on the construction of some nonlinear functionals was proved to be robust in the study of the well-posedness theories of hyperbolic conservation laws, especially in one space dimensional case. In particular, a generalized entropy functional was constructed in [T.-P. Liu, T. Yang, A new entropy functional for scalar conservation laws, Comm. Pure Appl. Math. 52 (1999) 1427–1442] for the L^1 stability of weak solutions. However, this generalized functional is so far only defined for scalar equations with convex flux function. In this paper, we introduce a new nonlinear functional which is motivated by the new Glimm functional introduced in [J.-L. Hua, Z.-H. Jiang, T. Yang, A new Glimm functional and convergence rate of Glimm scheme for general systems of hyperbolic conservation laws, preprint] for general scalar conservation laws. This functional improves the one given in [H.-X. Liu, T. Yang, A nonlinear functional for general scalar hyperbolic conservation laws, J. Differential Equations 235 (2) (2007) 658–667] and it can be viewed as a better attempt for the generalized entropy functional for general equations.

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1. Introduction

Consider the Cauchy problem for a scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where the flux function $f : R \mapsto R$ is C^1 and the initial data $u_0(x)$ satisfies some conditions imposed later. It is well known that even for smooth initial data, discontinuities, called shock waves usually form in finite time because of the nonlinearity in the flux function $f(u)$. So far the mathematical theories for the one space dimensional conservation laws have been well developed, cf. [1,2,4–8,10,12,19,20] and references therein. On the other hand, there are still some interesting and unsolved problems for this type of fundamental equations. For example, when one considers the well-posedness theory for the systems of hyperbolic conservation laws, the stability should be considered in the L^1 space. For this, a generalized entropy functional, now called Liu–Yang functional was constructed in [17] for scalar conservation laws with convex flux function. This generalized entropy functional is different from the classical entropy functional and it gives the control of the nonlinearity in each genuinely nonlinear characteristic fields for the study of the evolution of distance between two solutions in L^1 topology. However, so far this intrinsic functional is defined only to the case with convex flux function corresponding to the system whose characteristic fields are either genuinely nonlinear or linearly degenerate. Attempts have been made to define the generalized entropy functional for general scalar conservation laws, cf. [14]. And this paper also serves for this purpose, that is, we will introduce a new nonlinear functional for general scalar conservation laws and we will show that it gives a better dissipation rate than the previous one. However, it is still not good enough for the construction of a nonlinear functional in the study of L^1 stability for systems. Nevertheless, it improves the previous results and gives some new estimates which may be useful for the future investigation in this direction.

For later presentation, let us introduce some basic concepts of the weak solutions to the hyperbolic conservation laws. Firstly, the weak solution considered is defined as follows.

Definition 1.1. A function $u(x,t) : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a weak solution of the problem (1.1), if $u(x,t)$ is a bounded measurable function and

$$\iint\limits_{t\geqslant 0} \big[u(x,t)\phi_t(x,t) + f\big(u(x,t)\big)\phi_x(x,t)\big]\,dx\,dt + \int\limits_{t=0} u_0(x)\phi(x,0)\,dx = 0 \tag{1.2}$$

holds for any smooth function $\phi(x,t)$ with compact support in $\mathbb{R}^2$.

The existence of solutions to (1.1) can be proved by various methods such as the vanishing viscosity limit and difference schemes like Lax scheme, Godunov scheme, Glimm scheme and the wave front tracking. In both the Glimm scheme and the wave front tracking method, the solutions to the Riemann problem are used as building blocks. Here, Riemann problem means that the Cauchy problem has the initial data given by

$$u_0(x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \tag{1.3}$$

where $u^\pm$ are constants.

Since the weak solution is not unique and one needs to apply the physical entropy condition to choose the physical shock. For general scalar conservation laws, the following entropy condition was introduced in [19].

Definition 1.2. A discontinuity (u_-, u_+) is called an entropy shock if $\sigma(u_-, u) \geqslant \sigma(u_-, u_+)$ for all u between u_- and u_+.

In fact, the existence, uniqueness and continuous dependence of the entropy solution to scalar conservation laws were proved in the classical paper by Kruzkov as stated in the following.

Theorem 1.1 *(Kruzkov). Let $f : R \mapsto R$ be locally Lipschitz continuous. For each $u_0(x) \in L^\infty(\mathbb{R})$, the Cauchy problem* (1.1) *admits a unique global entropy solution.*

Moreover, we have the following L^1_{loc} stability estimate. Let $u_i(x,t)$ $(i = 1, 2)$ be the unique global entropy solution to (1.1) *with initial data $u_{i0}(x)$, then for each $R > 0$, we have for any $t > s \in [0,\infty) \setminus (E_0^{u_1} \cup E_0^{u_2})$*

$$\int\limits_{|x|\leqslant R} \big|u_1(x,t) - u_2(x,t)\big|\,dx \leqslant \int\limits_{|x|\leqslant R+d(t-s)} \big|u_1(x,s) - u_2(x,s)\big|\,dx. \tag{1.4}$$

Here

$$d = \sup_{|u|,|v|\leqslant M} \left|\frac{f(u) - f(v)}{u - v}\right|, \quad M = \max\big\{\big\|u_{10}(x)\big\|_{L^\infty(\mathbb{R})}, \big\|u_{20}(x)\big\|_{L^\infty(\mathbb{R})}\big\},$$

and $E_0^{u_1}$ and $E_0^{u_2}$ are measurable sets on $[0,\infty)$ with measure zero.

As mentioned before, even though the convex entropy functional is useful in the study of the L^1 perturbation of a constant state for systems of conservation laws, cf. [16], it is not suitable for the study of the L^1 distance between two weak solutions. In the framework of solutions with bounded total variation, the stability in L^1 norm was obtained in [1,3,4,15] and some references therein. A generalized entropy functional was introduced in [17] for scalar conservation laws with convex flux function and this functional captures exactly the nonlinear effect of each genuinely nonlinear characteristic field in the time evolution of solutions to systems of conservation laws, cf. [4,17,18]. In fact, the main purpose of introducing this entropy functional is to obtain the following key estimate. For any entropy solutions $u(x,t)$ and $v(x,t)$ with bounded total variations, $u_x(x,t)$ and $v_x(x,t)$ are distributions which may contain Dirac masses in connection with shock discontinuities. On the other hand, at each point x both $u(x\pm,t)$ and $v(x\pm,t)$ are well defined. As stated below, the entropy solutions $u(x,t)$ and $v(x,t)$ can be approximated by the partitions in either the deterministic version of the Glimm scheme or the wave front tracking scheme. In either case, the approximate solutions still denoted by $u(x,t)$ and $v(x,t)$ without any ambiguity are piecewise constant functions with bounded total variations. And each discontinuity in the solution denotes a wave which is either an entropy shock or a small rarefaction shock. Let $\Phi(u)$ and $\Phi(v)$ be the sets of all the waves in $u(x,t)$ and $v(x,t)$ at given time t, and $\{\alpha_k\}$ and $\{\beta_k\}$ denote the partitions of α and β which will be defined later in the introduction, respectively. With these notations, the key estimate takes the form

$$\begin{aligned}&\sum_{\alpha\in\Phi(u)\cup\Phi(v)} \sum_k \int\limits_0^t |\alpha_k|\big|u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)\big|\big|\sigma(\alpha_k) - \sigma\big(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)\big|\,dt\\ &\quad\leqslant O(1)\big(T.V.(u_0) + T.V.(v_0)\big)\|u_0 - v_0\|_{L^1}.\end{aligned} \tag{1.5}$$

Here, x_α denotes the location of a wave α, $\sigma(a,b) = \frac{f(a)-f(b)}{a-b}$ for $a \neq b$ is the speed of a wave or a virtual wave with a and b being the left and right states, respectively. When $a = b$, we use $\sigma(a)$ to denote $\sigma(a,b)$ which is obtained as a limit of the speed of an infinitesimal wave, namely $\sigma(a) = f'(a)$. Here, $T.V.(u_0)$ is the total variation of u_0 in x and $\|u_0 - v_0\|_{L^1}$ is the L^1 norm of $u_0 - v_0$ also in x.

The above estimate was proved to be true by the construction of the generalized entropy functional when the flux function is convex in [17], however, it remains unsolved for the general case. On the other hand, an attempt was made in [14] where a weaker estimate was proved by constructing a nonlinear functional for general scalar conservation laws. By the motivation of the new Glimm functional introduced in [9], in this paper, we will construct another new functional and prove another estimate which is still weaker than (1.5), but is stronger than those in the previous works.

For later use, we state some known properties of solutions to the general scalar conservations. Firstly, the solution operator of a scalar conservation law is L^1 contractive as stated in the following lemma, cf. [11].

Lemma 1.1. *Let $u_i(x,t)$, $i=1,2$, be two solutions of* (1.1) *satisfying the entropy condition, then for any $s \leqslant t$,*

$$\|u_1(x,t)-u_2(x,t)\|_{L^1} \leqslant \|u_1(x,s)-u_2(x,s)\|_{L^1}.$$

When we consider the interaction of two waves, it involves three states, denoted by u_i, $i=1,2,3$. For this, the quantity $A(u_1,u_2,u_3)$ denoting the area of the triangle bounded by the straight lines connecting points $(u_i, f(u_i))$, $i=1,2,3$, on the plane for the function $y=F(u)$ plays an important role. In fact, this quantity also relates to the bifurcation of the rarefaction wave curve from the Hugoniot curve in a corresponding system, cf. [16]. In the following, we will use this quantity in some situations. Note that

$$|u_1-u_2||u_1-u_3|\big|\sigma(u_1,u_2)-\sigma(u_1,u_3)\big| = sA(u_1,u_2,u_3),$$

for some positive constants s depending on the flux function $f(u)$ which remains uniformly bounded when u_1,u_2,u_3 range over a compact set.

Furthermore, in the study of wave interactions in the same family, as in [13], the following effective angle between waves α and β of the same ith family was introduced,

$$\Theta(\alpha,\beta) \equiv \theta_\alpha^+ + \theta_\beta^- + \sum \theta_\gamma.$$

θ_α^+ represents the value of λ_i at the right state of α minus its wave speed if α is a shock and is set to be zero if it is a rarefaction wave. Here, λ_i represents the characteristic of the ith family. Similarly the term θ_β^- denotes the difference between the speed of β and the value of λ_i at its left end state. θ_γ is the value of λ_i at the right state of the wave γ minus that of the left state. The summation $\sum\theta_\gamma$ is over the waves γ between α and β. When $\Theta(\alpha,\beta)$ is positive, the two waves will unlikely to meet; when $\Theta(\alpha,\beta)$ is negative, the two waves may eventually meet and interact. For scalar conservation laws, this effective angle between waves becomes simply the difference of wave speeds, that is, if α is on the left of β, then

$$-\Theta(\alpha,\beta) = \sigma(\alpha)-\sigma(\beta).$$

As in the deterministic version of Glimm scheme [13] for systems, all the waves in the solution can be partitioned into small subwaves as follows. In the following discussion, we will assume that a rarefaction wave is divided into several small rarefaction shocks with strength as the grid size of the Glimm scheme or a pre-chosen small constant in the wave front tracking method. In this way, the shock waves and rarefaction waves can be treated the same and the error thus caused tends to zero as the small constant approaches to zero. And this kind of partition also holds for the waves in the wave front tracking method.

More precisely, let the left state u_l be connected to the right state u_r by shocks (u_{j-1},u_j), and rarefaction waves (u_j,u_{j+1}), j odd, $1\leqslant j\leqslant m-1$, $u_0=u_l$ and $u_m=u_r$. A set of real numbers $\{v_0,v_1,\dots,v_p\}$ with $|v_i-v_{i+1}|\leqslant\epsilon$ for some pre-chosen small positive constant ϵ, $0\leqslant i\leqslant p-1$, is called a partition of (u_l,u_r) if

(i) $v_0 = u_l$, $v_p = u_r$, $v_{k-1} \leqslant v_k$, $k = 1, 2, \ldots, p$,
(ii) $\{u_0, u_1, \ldots, u_m\} \subset \{v_0, v_1, \ldots, v_p\}$.

With this partition, we can set

(1) $y_k \equiv v_k - v_{k-1}$,
(2) $\lambda_k \equiv \lambda(v_{k-1})$ and $[\lambda]_k \equiv [\lambda](v_{k-1}, v_k) \equiv \lambda(v_k) - \lambda(v_{k-1}) > 0$ if j is odd,
(3) $\lambda_k \equiv \sigma(u_{j-1}, u_j)$ and $[\lambda]_k \equiv [\lambda](v_{k-1}, v_k) \equiv 0$ if j is even.

For scalar conservation laws, the wave interaction involves only combination if the waves are in the same direction or cancellation if their directions are opposite. Let u_l, u_m and u_r be three states. Let α be the wave solving the Riemann problem (u_l, u_m) and denote its partition by $\alpha = \sum_{k=1}^{n_\alpha} \alpha_k$. The same notations apply to the waves β and γ solving the Riemann problems (u_l, u_m) and (u_m, u_r), respectively. Here, rarefaction waves are divided into small rarefaction shocks. Then we have

$$\gamma = \alpha + \beta, \tag{1.6}$$

$$\eta(\gamma) = \eta(\alpha) + \eta(\beta) + O(1)\big(|\alpha| + |\beta|\big)\epsilon, \tag{1.7}$$

with

$$\alpha = \sum_{k=1}^{n_\alpha} \alpha_k = u_m - u_l, \quad \beta = \sum_{k=1}^{n_\beta} \beta_k = u_r - u_m, \quad \text{and} \quad \gamma = \sum_{k=1}^{n_\gamma} \gamma_k = u_r - u_l,$$

$$\eta(\alpha) = \sum_{k=1}^{n_\alpha} \eta(\alpha_k), \quad \text{with } \eta(\alpha_k) = \alpha_k \lambda_k,$$

the same notations for $\eta(\beta)$ and $\eta(\gamma)$.

For later use, set the cancellation involved in the interaction by

$$C(u_l, u_m, u_r) \equiv \frac{1}{2}\big||\gamma| - |\alpha| - |\beta|\big|.$$

We now complete the brief introduction of the problem and the concepts and notations for later discussion. The rest of the paper will be organized as follows. In the next section, the new functional is defined and the main estimates in the paper are given. The non-increasing in time property of the component $\mathscr{F}(t)$ in the new functional corresponding to the wave interactions and wave crossings will be proved in Section 3. The non-increasing property of the new functional itself will be proved in Section 4 together with the proof of the main estimate.

2. Definition of $E(u, v)(t)$

In this section, we will give the definition of a new nonlinear functional $E(u, v)(t)$ and state the main results in this paper. In the following discussion, we assume that the weak solution of the Cauchy problem with initial data having bounded total variation is constructed by the wave front tracking method. Without loss of generality and for simplicity, we further assume that the initial data is piecewise constant. It is standard to show that the general case when the initial data has bounded variation can be taken as a limit of this case. Under this assumption, the solution contains finitely many discontinuities which are either entropy shocks or small rarefaction shocks with strength bounded by a pre-chosen small constant $\epsilon > 0$. And for the simplicity of notations, we will neglect the error of order $O(1)\epsilon$ which tends to zero as ϵ approaches to zero.

Let $u = u(x, t)$ and $v = v(x, t)$ be two such solutions to the scalar conservation law (1.1) with the initial data $u(x, 0)$ and $v(x, 0)$, respectively, satisfying $u(x, 0) - v(x, 0) \in L^1(\mathbb{R})$. As in [14], define

$$L(u,v)(x,t)=\begin{cases}\int_x^\infty (u-v)_{\operatorname{sign}(u-v)(x+)}(y,t)\,dy+\int_{-\infty}^x (u-v)_{-\operatorname{sign}(u-v)(x+)}(y,t)\,dy,\\ \quad \text{if } \sigma(u(x+,t),u(x-,t))\geqslant \sigma(u(x+,t),v(x+,t)),\\ \int_x^\infty (u-v)_{-\operatorname{sign}(u-v)(x+)}(y,t)\,dy+\int_{-\infty}^x (u-v)_{\operatorname{sign}(u-v)(x+)}(y,t)\,dy,\\ \quad \text{if } \sigma(u(x+,t),u(x-,t))<\sigma(u(x+,t),v(x+,t)),\end{cases}$$

where the positive and negative parts of $u-v$ are defined as $(u-v)_\pm=\frac{|u-v|\pm(u-v)}{2}$.

For any fixed partition of waves in the wave front tracking scheme, we define the new nonlinear functional in this paper by

$$\begin{aligned}E(u,v)(t)={}&\sum_{\alpha\in\Phi(u)}\sum_k B\big(u(x_{\alpha_k}-,t),u(x_{\alpha_k}+,t),v(x_{\alpha_k}+,t)\big)L(u,v)(x_{\alpha_k},t)\\&+\sum_{\alpha\in\Phi(v)}\sum_k B\big(v(x_{\alpha_k}-,t),v(x_{\alpha_k}+,t),u(x_{\alpha_k}+,t)\big)L(v,u)(x_{\alpha_k},t)\\&+K\mathscr{F}(t)\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1},\end{aligned}$$

where again $\Phi(u)$ and $\Phi(v)$ denote the sets of all the waves in u and v at given time t, and $\{\alpha_k\}$ and $\{\beta_k\}$ denote the partitions of α and β, respectively. Moreover,

$$B\big(u(x_{\alpha_k}-,t),u(x_{\alpha_k}+,t),v(x_{\alpha_k}+,t)\big)=\frac{|\alpha_k||q^{\alpha_k+}||\sigma(\alpha_k)-\sigma(u(x_\alpha+),v(x_\alpha+))|}{V(t)},$$

$$\mathscr{F}(t)=K_1V^2(t)+Q(u)(t)+Q(v)(t)+Q(u,v)(t).$$

Here

$$q^{\alpha+}=u(x_\alpha+,t)-v(x_\alpha+,t),\qquad V(t)=L(u)(t)+L(v)(t),$$

$$L(u)(t)=\sum_{\alpha\in\Phi(u)}|\alpha|,\qquad L(v)(t)=\sum_{\alpha\in\Phi(v)}|\alpha|,$$

$$Q(u,v)(t)=\sum_{\alpha\in\Phi(u),\,\beta\in\Phi(v)}\frac{\sum_{k,l}|\alpha_k||\beta_l||\sigma(\alpha_k)-\sigma(\beta_l)|}{T.V.(\alpha,\beta)}\chi\big((x_{\alpha_k}-x_{\beta_l})(\sigma(\beta_l)-\sigma(\alpha_k))\big),$$

$$Q(u)(t)=\sum_{\alpha,\beta\in\Phi(u)}\frac{\sum_{k,l}|\alpha_k||\beta_l||\sigma(\alpha_k)-\sigma(\beta_l)|}{T.V.(\alpha,\beta)}\chi\big((x_{\alpha_k}-x_{\beta_l})(\sigma(\beta_l)-\sigma(\alpha_k))\big),$$

with

$$T.V.(\alpha,\beta)=\sum\big\{|\gamma|:\ \gamma \text{ any wave in } u \text{ or } v \text{ and between } \alpha \text{ and } \beta, \text{ including } \alpha \text{ and } \beta\big\}.$$

Notice that even though the partitions of waves are not uniquely defined, one can check that the functional defined above is independent of the choice of the partitions because the difference tends to zero in the limit of the approximate solutions to the entropy solutions. From the theory of scalar conservation laws we also know that $V(t)$ is non-increasing in t.

Remark 2.1. In the definition of $E(u,v)(t)$, the component $Q(u,v)(t)$ is needed to control the jump in $B(\cdot,\cdot,\cdot)$ due to the crossing of two waves in u and v without wave interaction, cf. Figs. 6 and 7.

Remark 2.2. In the definition of $E(u,v)(t)$, we use the partition of the waves and take summation over them. If u and v are any two entropy solutions with bounded total variations, the summations can be replaced by integrations and the same estimates given in the following theorems also hold.

We are now ready to state the main results in this paper. The first two results are about the non-increasingness of the two nonlinear functionals $\mathscr{F}(t)$ and $E(u,v)(t)$. In fact, the functional $\mathscr{F}(t)$ is basically the Glimm functional except it also includes the effect of wave crossings. Notice that it is constant except at times of wave interaction and wave crossing.

Theorem 2.3. *$\mathscr{F}(t)$ is non-increasing at the times of the interaction and wave crossing if the total variation of the two weak solutions $u(x,t)$ and $v(x,t)$ are bounded.*

By using the non-increasingness of the functional $\mathscr{F}(t)$, the non-increasingness of the new nonlinear functional $E(u,v)(t)$ can be proved as stated in the following theorem.

Theorem 2.4. *Let $u(x,t)$, $v(x,t)$ be the weak solutions in Theorem* 2.3. *For the nonlinear functional $E(u,v)(t)$ defined above, there exist constants K, K_1 which may be suitably large, such that $E(u,v)(t)$ is non-increasing at the times of wave interaction and wave crossing.*

Finally, we can prove the following theorem which shows that the time derivative of the nonlinear functional $E(u,v)(t)$ except the points of wave interactions and wave crossings can be used to control the left-hand side of the key estimate (1.5). However, the upper bound thus obtained is larger than the expected upper bound given in (1.5).

Theorem 2.5. *Let $u(x,t)$, $v(x,t)$ be the two solutions in Theorem* 2.4*, we have*

$$\sum_{\alpha\in\Phi(u)\cup\Phi(v)}\sum_k\int_0^t |\alpha_k|\big|u(x_{\alpha_k}+,t)-v(x_{\alpha_k}+,t)\big|\big|\sigma(\alpha_k)-\sigma\big(u(x_{\alpha_k}+,t),v(x_{\alpha_k}+,t)\big)\big|\,dt$$
$$\leqslant CV(0)^{\frac{3}{2}}\|u_0-v_0\|_{L^1}^{\frac{1}{2}}\left[\int_0^t V(t)\,dt\right]^{\frac{1}{2}}.$$

The proof of Theorem 2.3 will be given in the next section. And Theorems 2.4 and 2.5 will be proved in Section 4. For later use, we now introduce a few more notations. Firstly, for a functional $H(t)$ on $u(x,t)$ and/or $v(x,t)$, we use $H(t^+)$ and $H(t^-)$ to denote the values of $H(t)$ before and after some given time t which is usually the time when wave interaction or wave crossing occurs. In addition, $\overline{H}(t)$ is the part in $H(t)$ which remains unchanged at time t through wave interaction or wave crossing. For brevity, sometimes we use $(\sigma(\alpha)-\sigma(\beta))^+$ to denote $|\sigma(\alpha)-\sigma(\beta)|\chi((x_\alpha-x_\beta)(\sigma(\beta)-\sigma(\alpha)))$, that is, if α is on the left of β, then

$$\big(\sigma(\alpha)-\sigma(\beta)\big)^+=\begin{cases}\sigma(\alpha)-\sigma(\beta), & \text{if } \sigma(\alpha)-\sigma(\beta)>0,\\ 0, & \text{otherwise.}\end{cases}$$

3. Estimation on $\mathscr{F}(t)$

In this section, we will prove that the nonlinear functional $\mathscr{F}(t)$ is non-increasing in time. In fact, it is basically the Glimm functional for wave interactions in a single solution except that it also captures the wave crossings between waves in two solutions. The control on the wave crossing is not needed for existence, however, it is needed to show that the new nonlinear functional $E(u,v)(t)$ is non-increasing in time.

Proof of Theorem 2.3. From the definition of $\mathscr{F}(t)$, it is obvious that it remains unchanged except when either wave interactions or wave crossings occur. Thus, in the following, we only need to estimate the change of $\mathscr{F}(t)$ at the time of interaction and crossing. Typically, there are only two cases

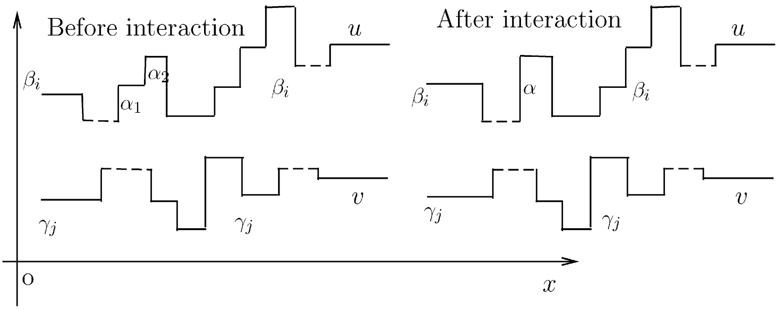

Fig. 1. Wave combination.

for wave interactions, that is, waves combination and wave cancellation for scalar conservation laws; and there is only one case when one wave in a solution crosses another wave in the other solution. These three cases will be discussed in details as follows.

Case 1. We first consider the case when wave combination occurs at the interaction time, that is, the two interacting waves involved are in the same direction, cf. Fig. 1.

Without loss of generality, we assume that the interacting waves before the interaction time t are $\alpha_1, \alpha_2 \in \Phi(u)$ and their combination yields α as shown in Fig. 1, where $\sigma(\alpha_2) < \sigma(\alpha_1)$ and α_1 is on the left of α_2. Furthermore, we can assume both α_1 and α_2 are positive. The other waves in the two solutions are denoted by $\beta_i \in \Phi(u), \gamma_j \in \Phi(v)$, where $i \in I$, $j \in J$ with I, J being some sets of indice. Without loss of generality, in the following estimation, we assume all the waves β_i, γ_j are all on the right of $\alpha, \alpha_1, \alpha_2$ because other cases can be discussed similarly.

By the definition of $Q(u)(t)$, we have

$$Q(u)\big(t^-\big) = \frac{\alpha_1\alpha_2(\sigma(\alpha_1) - \sigma(\alpha_2))}{\alpha_1 + \alpha_2} + \sum_{i \in I}\left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1) - \sigma(\beta_i))^+}{T.V.(\alpha_1, \beta_i)} + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2) - \sigma(\beta_i))^+}{T.V.(\alpha_2, \beta_i)}\right] + \overline{Q}(u)(t),$$

and

$$Q(u)\big(t^+\big) = \sum_{i \in I}\frac{\alpha|\beta_i|(\sigma(\alpha) - \sigma(\beta_i))^+}{T.V.(\alpha, \beta_i)} + \overline{Q}(u)(t).$$

Thus,

$$\Delta Q(u)(t) = Q(u)\big(t^+\big) - Q(u)\big(t^-\big) = -\frac{\alpha_1\alpha_2(\sigma(\alpha_1) - \sigma(\alpha_2))}{\alpha_1 + \alpha_2} + \sum_{i \in I}\left\{\frac{\alpha|\beta_i|(\sigma(\alpha) - \sigma(\beta_i))^+}{T.V.(\alpha, \beta_i)} - \left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1) - \sigma(\beta_i))^+}{T.V.(\alpha_1, \beta_i)} + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2) - \sigma(\beta_i))^+}{T.V.(\alpha_2, \beta_i)}\right]\right\}.$$

To estimate $\Delta Q(u)(t)$, we need to discuss several cases. Since $\sigma(\alpha_2) < \sigma(\alpha) < \sigma(\alpha_1)$, it is straightforward to check that there are only four subcases depending on the value of $\sigma(\beta_i)$:

(a) $\sigma(\alpha_2) < \sigma(\alpha) < \sigma(\beta_i) < \sigma(\alpha_1)$,
(b) $\sigma(\alpha_2) < \sigma(\beta_i) < \sigma(\alpha) < \sigma(\alpha_1)$,
(c) $\sigma(\beta_i) < \sigma(\alpha_2) < \sigma(\alpha) < \sigma(\alpha_1)$,
(d) $\sigma(\alpha_2) < \sigma(\alpha) < \sigma(\alpha_1) < \sigma(\beta_i)$.

Here and in the following, for brevity, we do not consider that case when the wave speeds of two different waves are the same in the functional because the contribution from these two waves is zero. Let I_a, I_b, I_c, I_d be the subsets of β_i corresponding to the subcases (a), (b), (c), (d), respectively. Then, $I = I_a \cup I_b \cup I_c \cup I_d$. And we have

$$\begin{aligned}
&\sum_{i\in I}\left\{\frac{\alpha|\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \\
&\quad = \left(\sum_{i\in I_a}+\sum_{i\in I_b}+\sum_{i\in I_c}+\sum_{i\in I_d}\right)\left\{\frac{\alpha|\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)}\right.\right. \\
&\qquad \left.\left. + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \leqslant 0.
\end{aligned}$$

In fact, it is obvious that

$$\left(\sum_{i\in I_a}+\sum_{i\in I_c}+\sum_{i\in I_d}\right)\left\{\frac{\alpha|\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \leqslant 0.$$

In addition, when $i \in I_b$, we have

$$\begin{aligned}
&\sum_{i\in I_b}\left\{\frac{\alpha|\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{\alpha_1|\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{\alpha_2|\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \\
&\quad = \sum_{i\in I_b}\left[\frac{\alpha|\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)} - \frac{\alpha_1|\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\right] \\
&\quad = \sum_{i\in I_b}\frac{\alpha_2|\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)} \leqslant 0.
\end{aligned}$$

Hence,

$$\Delta Q(u)(t) = Q(u)\bigl(t^+\bigr) - Q(u)\bigl(t^-\bigr) \leqslant -\frac{\alpha_1\alpha_2(\sigma(\alpha_1)-\sigma(\alpha_2))}{\alpha_1+\alpha_2}.$$

Since the wave combination is for waves in the solution u, in this case, we have

$$\Delta Q(v)(t) = 0,$$

and

$$Q(u,v)\bigl(t^-\bigr) = \sum_{j\in J}\left[\frac{\alpha_1|\gamma_j|(\sigma(\alpha_1)-\sigma(\gamma_j))^+}{T.V.(\alpha_1,\gamma_j)} + \frac{\alpha_2|\gamma_j|(\sigma(\alpha_2)-\sigma(\gamma_j))^+}{T.V.(\alpha_2,\gamma_j)}\right] + \overline{Q}(u,v)(t),$$

$$Q(u,v)\bigl(t^+\bigr) = \sum_{j\in J}\frac{\alpha|\gamma_j|(\sigma(\alpha)-\sigma(\gamma_j))^+}{T.V.(\alpha,\gamma_j)} + \overline{Q}(u,v)(t).$$

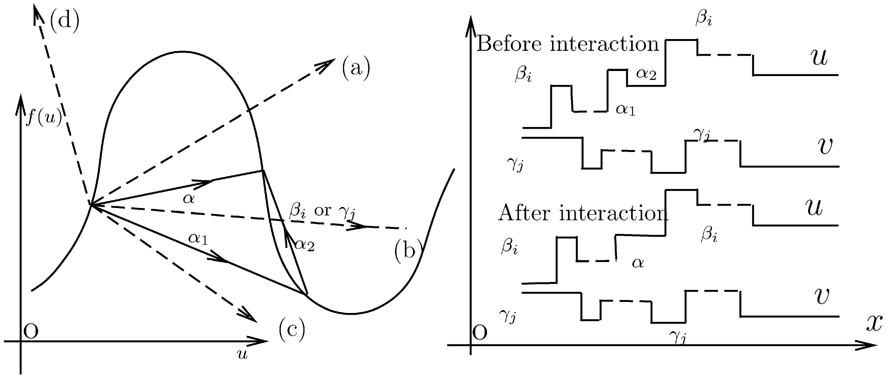

Fig. 2. Wave cancellation.

Therefore,

$$\Delta Q(u,v)(t) = Q(u,v)\big(t^+\big) - Q(u,v)\big(t^-\big)$$
$$= \sum_{j\in J}\left\{\frac{\alpha|\gamma_j|(\sigma(\alpha)-\sigma(\gamma_j))^+}{T.V.(\alpha,\gamma_j)} - \left[\frac{\alpha_1|\gamma_j|(\sigma(\alpha_1)-\sigma(\gamma_j))^+}{T.V.(\alpha_1,\gamma_j)} + \frac{\alpha_2|\gamma_j|(\sigma(\alpha_2)-\sigma(\gamma_j))^+}{T.V.(\alpha_2,\gamma_j)}\right]\right\}.$$

Similar to the argument for $\Delta Q(u)(t)$ given above, we can show that

$$\Delta Q(u,v)(t) \leqslant 0.$$

Now we can conclude that if the wave interaction involves only wave combinations, then

$$\Delta\mathscr{F}(t) \leqslant -\frac{\alpha_1\alpha_2(\sigma(\alpha_1)-\sigma(\alpha_2))}{\alpha_1+\alpha_2} < 0.$$

Case 2. We now consider the case when wave cancellation occurs, cf. Fig. 2.

Without loss of generality, we assume that $\alpha_1, \alpha > 0, \alpha_2 < 0$ so that at the interaction time t, $\alpha_1 + \alpha_2 \to \alpha$ with $\alpha_1 > |\alpha_2|$. In addition, $\sigma(\alpha_2) < \sigma(\alpha_1)$ and α_1 is on the left of α_2. As before, the other waves in the solutions are denoted by $\beta_i \in \Phi(u), \gamma_j \in \Phi(v)$ where $i \in I, j \in J$. Again, without loss of generality, we assume that the waves β_i, γ_j are on the right of $\alpha, \alpha_1, \alpha_2$.

Under the above assumptions, we have

$$\Delta V^2(t) = -2|\alpha_2|\big(V(t+) + V(t-)\big),$$

and

$$Q(u)\big(t^-\big) = \frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|} + \sum_{i\in I}\left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]$$
$$+ \overline{Q}(u)(t),$$

$$Q(u)\big(t^+\big) = \sum_{i\in I}\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} + \overline{Q}(u).$$

Hence,

$$\Delta Q(u)(t) = Q(u)\big(t^+\big) - Q(u)\big(t^-\big) = -\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|} + \sum_{i\in I}\left\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\}.$$

Similar to Case 1, we also have the following four subcases to consider the change of $Q(u)(t)$ and $Q(u,v)(t)$:

(a) $\sigma(\alpha_2) < \sigma(\alpha_1) < \sigma(\alpha) < \sigma(\beta_i)$,
(b) $\sigma(\alpha_2) < \sigma(\alpha_1) < \sigma(\beta_i) < \sigma(\alpha)$,
(c) $\sigma(\alpha_2) < \sigma(\beta_i) < \sigma(\alpha_1) < \sigma(\alpha)$,
(d) $\sigma(\beta_i) < \sigma(\alpha_2) < \sigma(\alpha_1) < \sigma(\alpha)$.

We claim that

$$\begin{aligned}
&\sum_{i\in I}\left\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \\
&\quad = \left[\sum_{i\in I_a}+\sum_{i\in I_b}+\sum_{i\in I_c}+\sum_{i\in I_d}\right]\left\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)}\right.\right. \\
&\qquad \left.\left. + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \\
&\quad = O(1)L(u)(t-)|\alpha_2|.
\end{aligned}$$

In fact,

$$\sum_{i\in I_a}\left\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} = 0.$$

Since

$$\begin{aligned}
&\sum_{i\in I_b}\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)} \\
&\quad = \sum_{i\in I_b}\left[\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)} + |\alpha||\beta_i|\big(\sigma(\alpha)-\sigma(\beta_i)\big)\left(\frac{1}{T.V.(\alpha,\beta_i)} - \frac{1}{T.V.(\alpha_1,\beta_i)}\right)\right] \\
&\quad \leqslant \sum_{i\in I_b}\left[\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\alpha_1))}{T.V.(\alpha_1,\beta_i)} + \frac{2|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\alpha_1))|\alpha_2|}{(T.V.(\alpha_1,\beta_i))(T.V.(\alpha,\beta_i))}\right] \leqslant O(1)L(u)(t-)|\alpha_2|,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{i\in I_b}\left\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)} - \left[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)} + \frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\right]\right\} \\
&\quad = \sum_{i\in I_b}\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)} = O(1)L(u)(t-)|\alpha_2|.
\end{aligned}$$

In addition, since

$$\sum_{i\in I_c}\bigg\{\bigg[\frac{1}{T.V.(\alpha,\beta_i)}-\frac{1}{T.V.(\alpha_1,\beta_i)}\bigg]|\alpha||\beta_i|\big(\sigma(\alpha)-\sigma(\beta_i)\big)$$
$$+\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))-|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\bigg\}$$
$$=\sum_{i\in I_c}\bigg[\frac{2|\alpha_2||\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)T.V.(\alpha_1,\beta_i)}+\frac{|\alpha_2||\beta_i|(\sigma(\beta_i)-\sigma(\alpha_2))}{T.V.(\alpha_1,\beta_i)}\bigg]$$
$$\leqslant\sum_{i\in I_c}\bigg[\frac{2|\alpha_2||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}+\frac{|\alpha_2||\beta_i|(\sigma(\beta_i)-\sigma(\alpha_2))}{T.V.(\alpha_1,\beta_i)}\bigg]$$
$$\leqslant\sum_{i\in I_c}2\bigg[\frac{|\alpha_2||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}+\frac{|\alpha_2||\beta_i|(\sigma(\beta_i)-\sigma(\alpha_2))}{T.V.(\alpha_1,\beta_i)}\bigg]=O(1)L(u)(t-)|\alpha_2|,$$

we obtain

$$\sum_{i\in I_c}\bigg\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)}-\bigg[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)}+\frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\bigg]\bigg\}$$
$$=\sum_{i\in I_c}\bigg[\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)}-\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\bigg]$$
$$=\sum_{i\in I_c}\bigg\{\bigg[\frac{1}{T.V.(\alpha,\beta_i)}-\frac{1}{T.V.(\alpha_1,\beta_i)}\bigg]|\alpha||\beta_i|\big(\sigma(\alpha)-\sigma(\beta_i)\big)$$
$$+\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))-|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\bigg\}$$
$$\leqslant O(1)L(u)(t-)|\alpha_2|.$$

Finally, we have

$$\sum_{i\in I_d}\bigg\{\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))^+}{T.V.(\alpha,\beta_i)}-\bigg[\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))^+}{T.V.(\alpha_1,\beta_i)}+\frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))^+}{T.V.(\alpha_2,\beta_i)}\bigg]\bigg\}$$
$$\leqslant\sum_{i\in I_d}\bigg[\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)}-\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}-\frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\bigg]$$
$$=O(1)L(u)(t-)|\alpha_2|.$$

Here we have used the fact that

$$\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)}-\frac{|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}-\frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}$$
$$=\frac{2|\alpha_2||\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha,\beta_i)T.V.(\alpha_1,\beta_i)}$$
$$+\frac{|\alpha||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))-|\alpha_1||\beta_i|(\sigma(\alpha_1)-\sigma(\beta_i))-|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}$$

$$\begin{aligned}&\leqslant 2\frac{|\alpha_2||\beta_i|(\sigma(\alpha)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}-2\frac{|\alpha_2||\beta_i|(\sigma(\alpha_2)-\sigma(\beta_i))}{T.V.(\alpha_1,\beta_i)}\\&=2\frac{|\alpha_2||\beta_i|(\sigma(\alpha)-\sigma(\alpha_2))}{T.V.(\alpha_1,\beta_i)},\end{aligned}$$

and that $|\sigma(\alpha)-\sigma(\alpha_2)|$, $|\sigma(\alpha)-\sigma(\alpha_1)|$, and $|\sigma(\alpha_1)-\sigma(\alpha_2)|$ can be controlled by $O(1)(|\alpha_1|+|\alpha_2|)$.

For $Q(u,v)(t)$, firstly, we have

$$\begin{aligned}Q(u,v)\bigl(t^-\bigr)=\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|}&+\sum_{j\in J}\biggl[\frac{|\alpha_1||\gamma_j|(\sigma(\alpha_1)-\sigma(\gamma_j))^+}{T.V.(\alpha_1,\gamma_j)}\\&+\frac{|\alpha_2||\gamma_j|(\sigma(\alpha_2)-\sigma(\gamma_j))^+}{T.V.(\alpha_2,\gamma_j)}\biggr]+\overline{Q}(u,v)(t),\end{aligned}$$

and

$$Q(u,v)\bigl(t^+\bigr)=\sum_{j\in J}\frac{|\alpha||\gamma_j|(\sigma(\alpha)-\sigma(\gamma_j))^+}{T.V.(\alpha,\gamma_j)}+\overline{Q}(u,v)(t).$$

Hence,

$$\begin{aligned}\Delta Q(u,v)(t)=Q(u,v)\bigl(t^+\bigr)-Q(u,v)\bigl(t^-\bigr)=\sum_{j\in J}\biggl\{&\frac{|\alpha||\gamma_j|(\sigma(\alpha)-\sigma(\gamma_j))^+}{T.V.(\alpha,\gamma_j)}\\&-\biggl[\frac{|\alpha_1||\gamma_j|(\sigma(\alpha_1)-\sigma(\gamma_j))^+}{T.V.(\alpha_1,\gamma_j)}+\frac{|\alpha_2||\gamma_j|(\sigma(\alpha_2)-\sigma(\gamma_j))^+}{T.V.(\alpha_2,\gamma_j)}\biggr]\biggr\}.\end{aligned}$$

Similar to the argument for $\Delta Q(u)(t)$, we can obtain

$$\Delta Q(u,v)(t)\leqslant O(1)L(v)(t-)|\alpha_2|.$$

In conclusion, for Case 2, by choosing the coefficient K_1 in $\mathscr{F}(t)$ sufficiently large, we have that there exists a constant $C>0$ such that

$$\Delta\mathscr{F}\leqslant-\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|}-CV(t-)|\alpha_2|<0.$$

Case 3. Finally, we turn to the case when there is a wave in $u(x,t)$ crosses another wave in $v(x,t)$ at time t. To be definite, assume that there is a wave of $\alpha_1\in\Phi(u)$ and wave of $\alpha_2\in\Phi(v)$ with α_2 on the right of α_1 and $\sigma(\alpha_1)>\sigma(\alpha_2)$. And at time t, α_1 crosses α_2 as shown in Fig. 3. It is easy to see that

$$\Delta Q(u)(t)=\Delta Q(v)(t)=0,$$

and

$$\Delta Q(u,v)(t)=-\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|}.$$

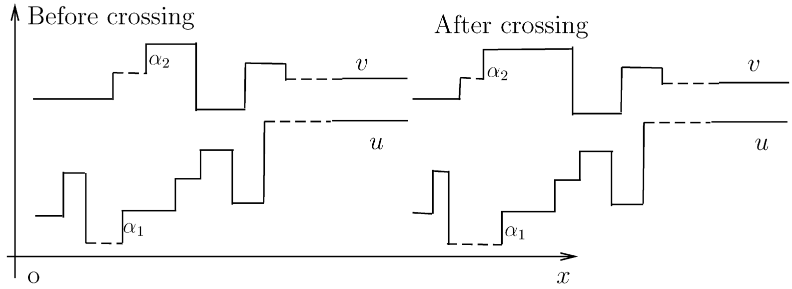

Fig. 3. Wave crossing.

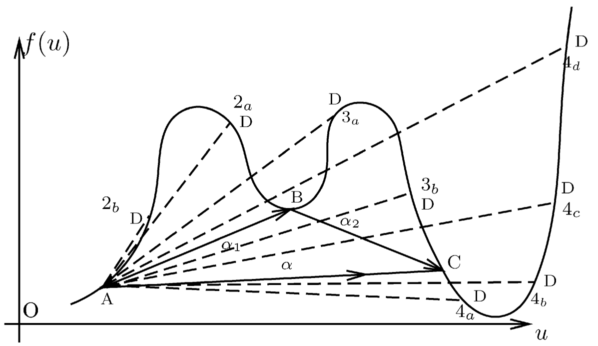

Fig. 4. Case 1 of Theorem 2.4.

Thus,

$$\Delta\mathscr{F}(t) = -\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1) - \sigma(\alpha_2))}{|\alpha_1| + |\alpha_2|} < 0.$$

Therefore, this completes the proof of Theorem 2.3. $\square$

4. Estimation on $E(u, v)(t)$

In this section, we will first prove Theorem 2.4 about the decrease of the new functional $E(u, v)(t)$ at the times of wave interactions and wave crossings. Since it involves the areas on the left or on the right of a wave at time t, that is, $L(u, v)(x, t^{\pm})$ at some location x, the proof is more delicate than the one for $\mathscr{F}(t)$. After that, we will differentiate the functional $E(u, v)(t)$ between the times of wave interactions and crossings to derive the desired estimate stated in Theorem 2.5.

Proof of Theorem 2.4. Similar to the proof of Theorem 2.3, we also need to discuss three main cases for the functional $E(u, v)(t)$, however, there are more subcases.

Case 1. Under the same assumption as Case 1 in the proof of Theorem 2.3 for wave combination, cf. Figs. 1 and 4.

Without loss of generality, we take $u_A < u_B < u_C$. According to the location of v_D and the definition of $L(u, v)$, we have the following four subcases. And in each subcase, there are also several cases which will be discussed as follows.

Subcase (1). Assume $v_D < u_A < u_B < u_C$. Then we have the following possibilities:

$$1_a.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$1_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$1_c.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D),$$

$$1_d.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D).$$

Subcase (2). Assume $u_A < v_D < u_B < u_C$. By the entropy condition, we know that there are only two possibilities under this condition:

$$2_a.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$2_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D).$$

Subcase (3). Assume $u_A < u_B < v_D < u_C$. Similar to the subcase (2), we have

$$3_a.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$3_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D).$$

Subcase (4). Assume $u_A < u_B < u_C < v_D$. Then we have the following possibilities:

$$4_a.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$4_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$4_c.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D),$$

$$4_d.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D).$$

For illustration, we will give the estimation on the subcases 3_a and 4_c, the other cases can be considered similarly.

Firstly, by the definition of the nonlinear functional, we have

$$\begin{aligned} E(u,v)\big(t^-\big) &= B(u_A, u_B, v_D)L(u,v)\big(x_{\alpha_1}, t^-\big) + B(u_B, u_C, v_D)L(u,v)\big(x_{\alpha_2}, t^-\big) \\ &\quad + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^-\big)\big\|u(.,t) - v(.,t)\big\|_{L^1}, \end{aligned}$$

and

$$E(u,v)\big(t^+\big) = B(u_A, u_C, v_D)L(u,v)\big(x_\alpha, t^+\big) + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^+\big)\big\|u(.,t) - v(.,t)\big\|_{L^1}.$$

Thus,

$$\begin{aligned} \Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big) \\ &= B(u_A, u_C, v_D)L(u,v)\big(x_\alpha, t^+\big) - \big[B(u_A, u_B, v_D)L(u,v)\big(x_{\alpha_1}, t^-\big)\big] \\ &\quad + B(u_B, u_C, v_D)L(u,v)\big(x_{\alpha_2}, t^-\big) + K\Delta\mathscr{F}(t)\big\|u(.,t) - v(.,t)\big\|_{L^1}. \end{aligned}$$

Notice that in the subcase 4_c, $L(u,v)(x_\alpha,t^+) = L(u,v)(x_{\alpha_2},t^-)$ holds. Hence, by choosing K sufficiently large, we have

$$\begin{aligned}\Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\ &\leqslant \big[B(u_A,u_C,v_D) - B(u_B,u_C,v_D)\big]L(u,v)\big(x_{\alpha_2},t^-\big) + K\Delta\mathscr{F}\big\|u(.,t)-v(.,t)\big\|_{L^1} \leqslant 0.\end{aligned}$$

Here, we have used the fact that $V(t+) = V(t-) \geqslant |\alpha_1| + |\alpha_2|$, and

$$\begin{aligned}B(u_A,u_C,v_D) - B(u_B,u_C,v_D) &= s\left[\frac{A(u_A,u_C,v_D)}{V(t+)} - \frac{A(u_B,u_C,v_D)}{V(t-)}\right]\\ &\leqslant s\frac{A(u_A,u_B,u_C)}{V(t+)}\\ &\leqslant \frac{\alpha_1\alpha_2(\sigma(\alpha_1)-\sigma(\alpha_2))}{\alpha_1+\alpha_2},\end{aligned}$$

together with

$$\Delta\mathscr{F}(t) \leqslant -\frac{\alpha_1\alpha_2(\sigma(\alpha_1)-\sigma(\alpha_2))}{\alpha_1+\alpha_2}.$$

Notice that in the subcase 3_a, it holds that $(u-v)|_{x_{\alpha_1}+} < 0$, $(u-v)|_{x_\alpha+} > 0$, $(u-v)|_{x_{\alpha_2}+} > 0$, $\sigma(\alpha_1) < \sigma(u_B,v_D), \sigma(\alpha_2) > \sigma(u_C,v_D), \alpha(\alpha) > \sigma(u_C,v_D)$. By the definition of $L(u,v)$, we have

$$L(u,v)\big(x_\alpha,t^+\big) = \int_{x_\alpha}^{\infty}(u-v)_+(y,t)\,dy + \int_{-\infty}^{x_\alpha}(u-v)_-(y,t)\,dy,$$

$$L(u,v)\big(x_{\alpha_2},t^-\big) = \int_{x_{\alpha_2}}^{\infty}(u-v)_+(y,t)\,dy + \int_{-\infty}^{x_{\alpha_2}}(u-v)_-(y,t)\,dy,$$

$$L(u,v)\big(x_{\alpha_1},t^-\big) = \int_{x_{\alpha_1}}^{\infty}(u-v)_+(y,t)\,dy + \int_{-\infty}^{x_{\alpha_1}}(u-v)_-(y,t)\,dy.$$

Thus, $L(u,v)(x_\alpha,t^+) = L(u,v)(x_{\alpha_2},t^-) = L(u,v)(x_{\alpha_1},t^-)$. Hence, by choosing K sufficiently large, we also have

$$\begin{aligned}\Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\ &\leqslant \big[B(u_A,u_C,v_D) - B(u_B,u_C,v_D) - B(u_A,u_B,v_D)\big]L(u,v)\big(x_{\alpha_1},t^-\big)\\ &\quad + K\Delta\mathscr{F}\big\|u(.,t)-v(.,t)\big\|_{L^1}\\ &\leqslant 0.\end{aligned}$$

Here, we have used the fact that

$$B(u_A,u_C,v_D) - B(u_B,u_C,v_D) - B(u_A,u_B,v_D) = 0.$$

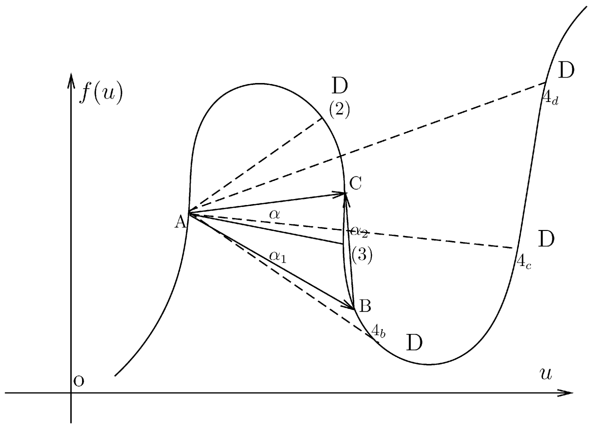

Fig. 5. Case 2 of Theorem 2.4.

Case 2. Corresponding to Case 2 in the proof of Theorem 2.3, we now consider the wave cancellation, cf. Figs. 2 and 5.

As in Case 1, we also take $u_A < u_C < u_B$ without loss of generality. According to the definition of $L(u, v)$ and the location of v_D have the following possibilities.

Subcase (1). When $v_D < u_A < u_C < u_B$, the following situations may occur

$$1_a.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$1_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$1_c.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$1_d.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D).$$

Subcase (2). When $u_A < v_D < u_C < u_B$, by the entropy condition, we have

$$\sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D).$$

Subcase (3). When $u_A < u_C < v_D < u_B$, similar to subcase (2), we have

$$\sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D).$$

Subcase (4). When $u_A < u_B < u_C < v_D$, we have the following possibilities:

$$4_a.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) > \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$4_b.\ \sigma(\alpha_1) > \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$4_c.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) > \sigma(u_C, v_D),$$

$$4_d.\ \sigma(\alpha_1) < \sigma(u_B, v_D), \qquad \sigma(\alpha_2) < \sigma(u_C, v_D), \qquad \sigma(\alpha) < \sigma(u_C, v_D).$$

Notice that the subcase 4_a can happen only when α_2 is a rarefaction shock. For illustration, in the following, we only give the estimation on the typical subcases 4_b, 4_c and 4_d. The other cases can be discussed similarly.

Firstly, notice again that

$$E(u,v)\big(t^-\big) = B(u_A,u_B,v_D)L(u,v)\big(x_{\alpha_1},t^-\big) + B(u_B,u_C,v_D)L(u,v)\big(x_{\alpha_2},t^-\big) + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^-\big)\big\|u(.,t)-v(.,t)\big\|_{L^1},$$

and

$$E(u,v)\big(t^+\big) = B(u_A,u_C,v_D)L(u,v)\big(x_\alpha,t^+\big) + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^+\big)\big\|u(.,t)-v(.,t)\big\|_{L^1}.$$

Hence,

$$\begin{aligned}\Delta E(u,v)(t) = E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big) &= B(u_A,u_C,v_D)L(u,v)\big(x_\alpha,t^+\big)\\ &\quad - \big[B(u_A,u_B,v_D)L(u,v)\big(x_{\alpha_1},t^-\big) + B(u_B,u_C,v_D)L(u,v)\big(x_{\alpha_2},t^-\big)\big]\\ &\quad + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}.\end{aligned}$$

For the subcase 4_b, we have $L(u,v)(x_\alpha,t^+) = L(u,v)(x_{\alpha_1},t^-) \neq L(u,v)(x_{\alpha_2},t^-)$, so that

$$\begin{aligned}\Delta E(u,v)(t) &\leqslant \big[B(u_A,u_C,v_D) - B(u_A,u_B,v_D)\big]L(u,v)\big(x_\alpha,t^+\big) + K\Delta\mathscr{F}\big\|u(.,t)-v(.,t)\big\|_{L^1}\\ &\leqslant 0.\end{aligned}$$

In fact,

$$\begin{aligned}&B(u_A,u_C,v_D) - B(u_A,u_B,v_D)\\ &\quad = s\left[\frac{A(u_A,u_C,v_D)}{V(t+)} - \frac{A(u_A,u_B,v_D)}{V(t-)}\right]\\ &\quad = s\left[\frac{2|\alpha_2|A(u_A,u_C,v_D)}{V(t+)V(t-)} + \frac{A(u_A,u_C,v_D) - A(u_A,u_B,v_D)}{V(t-)}\right]\\ &\quad \leqslant \frac{2|\alpha_2||u_C-v_D|(\sigma(\alpha)-\sigma(u_C,v_D))}{V(t-)}\\ &\qquad + \frac{|\alpha_2||\alpha_1|(\sigma(\alpha_1)-\sigma(\alpha_2)) + |\alpha_2||u_C-v_D|(\sigma(u_C,v_D)-\sigma(\alpha_2))}{V(t-)}\\ &\quad \leqslant \frac{2|\alpha_2||u_C-v_D|(\sigma(\alpha)-\sigma(\alpha_2))}{V(t-)} + \frac{|\alpha_2||\alpha_1|(\sigma(\alpha_1)-\sigma(\alpha_2)) + |\alpha_2||u_C-v_D|(\sigma(\alpha)-\sigma(\alpha_2))}{V(t-)}\\ &\quad = O(1)V(t-)|\alpha_2|.\end{aligned}$$

For the subcase 4_c, firstly notice that $L(u,v)(x_\alpha,t^+) \neq L(u,v)(x_{\alpha_1},t^-) = L(u,v)(x_{\alpha_2},t^-)$. Since

$$\begin{aligned}B(u_A,u_C,v_D) &= \frac{|\alpha||u_C-v_D|(\sigma(\alpha)-\sigma(u_C,u_D))}{V(t+)}\\ &= \frac{|\alpha||u_B-v_D|(\sigma(\alpha)-\sigma(u_B,u_D)) + |\alpha||\alpha_2|(\sigma(\alpha)-\sigma(\alpha_2))}{V(t+)}\\ &\leqslant \frac{|\alpha||u_B-v_D|(\sigma(\alpha)-\sigma(\alpha_1)) + |\alpha||\alpha_2|(\sigma(\alpha)-\sigma(\alpha_2))}{V(t+)}\end{aligned}$$

$$
\begin{aligned}
&= O(1)V(t-)|\alpha_2| + \frac{|\alpha||u_B - v_D|(\sigma(\alpha) - \sigma(\alpha_1))}{V(t-)} + \frac{|\alpha_2||\alpha||u_B - v_D|(\sigma(\alpha) - \sigma(\alpha_1))}{V(t-)V(t+)} \\
&= O(1)V(t-)|\alpha_2| + \frac{|\alpha_2||u_B - v_D|(\sigma(\alpha_1) - \sigma(\alpha_2))}{V(t-)} + O(1)V(t-)|\alpha_2| \\
&= O(1)V(t-)|\alpha_2|,
\end{aligned}
$$

and

$$
\Delta\mathscr{F}(t) \leqslant -\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1) - \sigma(\alpha_2))}{|\alpha_1| + |\alpha_2|} - CV(t-)|\alpha_2|,
$$

we have by choosing K sufficiently large that

$$
\begin{aligned}
\Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big) \\
&\leqslant B(u_A, u_C, v_D)L(u,v)\big(x_\alpha, t^+\big) + K\Delta\mathscr{F}(t)\big\|u(.,t) - v(.,t)\big\|_{L^1} \leqslant 0.
\end{aligned}
$$

For the subcase 4_d, we have $L(u,v)(x_\alpha, t^+) = L(u,v)(x_{\alpha_1}, t^-) = L(u,v)(x_{\alpha_2}, t^-)$. This gives that

$$
\begin{aligned}
\Delta E(u,v)(t) = {} & \big[B(u_A, u_C, v_D) - B(u_A, u_B, v_D) - B(u_B, u_C, v_D)\big]L(u,v)\big(x_{\alpha_1}, t^-\big) \\
& + K\Delta\mathscr{F}\big\|u(.,t) - v(.,t)\big\|_{L^1} \leqslant 0,
\end{aligned}
$$

because

$$
\begin{aligned}
& B(u_A, u_C, v_D) - B(u_A, u_B, v_D) - B(u_B, u_C, v_D) \\
& \quad = s\left[\frac{A(u_A, u_C, v_D)}{V(t+)} - \frac{A(u_A, u_B, v_D)}{V(t-)} - \frac{A(u_B, u_C, v_D)}{V(t-)}\right] \\
& \quad \leqslant s\frac{2|\alpha_2|A(u_A, u_C, v_D)}{V(t+)V(t-)} - s\frac{2A(u_B, u_C, v_D)}{V(t-)} \\
& \quad \leqslant \frac{2|\alpha_2||u_C - v_D|(\sigma(u_C, v_D) - \sigma(\alpha))}{V(t-)} - 2\frac{|\alpha_2||u_C - v_D|(\sigma(u_C, v_D) - \sigma(\alpha_2))}{V(t-)} \\
& \quad \leqslant \frac{2|\alpha_2||u_C - v_D|(\sigma(\alpha_2) - \sigma(\alpha))}{V(t-)} \leqslant 0.
\end{aligned}
$$

In the above calculation we have used the fact that $|u(x,t) - v(x,t)|$ can be controlled by $V(t)$ because $u(x,t) - v(x,t) \in L^1$.

Case 3. We now turn to the case with wave crossing as in Case 3 in the proof of Theorem 2.3. Without loss of generality, we assume $\alpha_1 = (u_A, u_B), \alpha_2 = (v_C, v_D)$ and $u_A < u_B < v_C < v_D$. According to the relative locations of α_1 and α_2, there are three typical cases given as follows. We know in all these subcases $V(t+) = V(t-) = V$, and

$$
\begin{aligned}
E(u,v)\big(t^-\big) = {} & B(u_A, u_B, v_C)L(u,v)\big(x_{\alpha_1}, t^-\big) + B(u_B, v_C, v_D)L(v,u)\big(x_{\alpha_2}, t^-\big) \\
& + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^-\big)\big\|u(.,t) - v(.,t)\big\|_{L^1},
\end{aligned}
$$

$$
\begin{aligned}
E(u,v)\big(t^+\big) = {} & B(u_A, u_B, v_D)L(u,v)\big(x_{\alpha_1}, t^+\big) + B(u_A, v_C, v_D)L(v,u)\big(x_{\alpha_2}, t^+\big) \\
& + \bar{E}(u,v)(t) + K\mathscr{F}\big(t^+\big)\big\|u(.,t) - v(.,t)\big\|_{L^1}.
\end{aligned}
$$

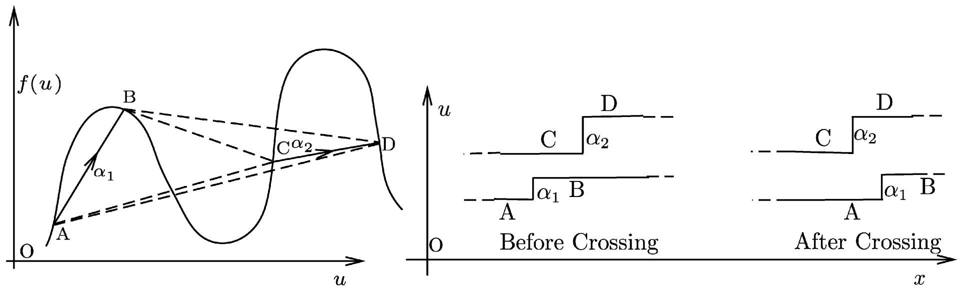

Fig. 6. Subcase (a) in Case 3 of Theorem 2.4.

Thus,

$$
\begin{aligned}
\Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\
&= B(u_A,u_B,v_D)L(u,v)\big(x_{\alpha_1},t^+\big) + B(u_A,v_C,v_D)L(v,u)\big(x_{\alpha_2},t^+\big)\\
&\quad - \big[B(u_A,u_B,v_C)L(u,v)\big(x_{\alpha_1},t^-\big) + B(u_B,v_C,v_D)L(v,u)\big(x_{\alpha_2},t^-\big)\big]\\
&\quad + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}.
\end{aligned}
$$

We are now going to discuss several typical cases.

Subcase (a). As shown in Fig. 6, the speed of $\alpha_1 \in \Phi(u)$ is larger than the one of $\alpha_2 \in \Phi(v)$, and α_1 lies on the left of α_2. Assume at time t, α_1 crosses α_2. In this case, firstly, we have

$$
L(u,v)\big(x_{\alpha_1},t^+\big) = L(u,v)\big(x_{\alpha_1},t^-\big) = L(v,u)\big(x_{\alpha_2},t^-\big) \neq L(v,u)\big(x_{\alpha_2},t^+\big).
$$

Hence, we obtain

$$
\begin{aligned}
&E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\
&\quad = s\bigg[\frac{A(u_A,u_B,v_D)L(u,v)(x_{\alpha_1},t^+)}{V} + \frac{A(u_A,v_C,v_D)L(v,u)(x_{\alpha_2},t^+)}{V} - \frac{A(u_A,u_B,v_C)L(u,v)(x_{\alpha_1},t^-)}{V}\\
&\qquad - \frac{A(u_B,u_C,u_D)L(v,u)(x_{\alpha_2},t^-)}{V}\bigg] + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}\\
&\quad = s\bigg\{\frac{[A(u_A,u_B,v_D) - A(u_A,u_B,v_C) - A(u_B,u_C,u_D)]L(u,v)(x_{\alpha_1},t^+)}{V}\\
&\qquad + \frac{A(u_A,v_C,v_D)L(v,u)(x_{\alpha_2},t^+)}{V}\bigg\} + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}\\
&\quad = s\bigg[\frac{A(u_A,v_C,v_D)L(u,v)(x_{\alpha_1},t^+)}{V} + \frac{A(u_A,v_C,v_D)L(v,u)(x_{\alpha_2},t^+)}{V}\bigg] + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}\\
&\quad = O(1)\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|}\big\|u(.,t)-v(.,t)\big\|_{L^1} + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1}\\
&\quad \leqslant 0,
\end{aligned}
$$

where K is chosen to be sufficiently large. Here, we have used the fact that

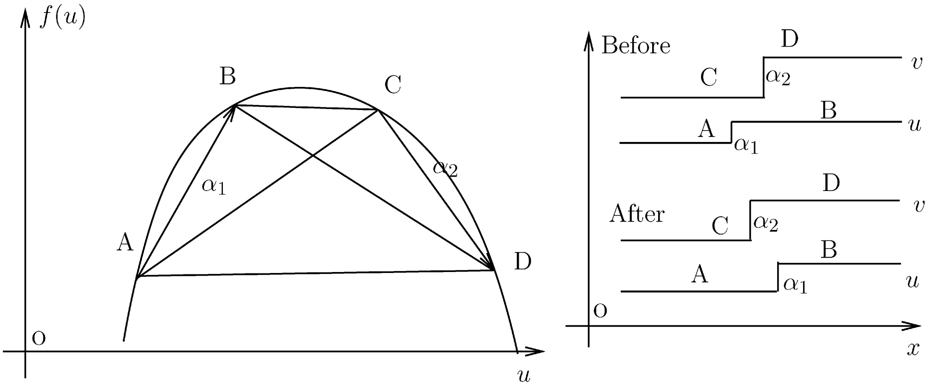

Fig. 7. Subcase (b) in Case 3 of Theorem 2.4.

$$\Delta\mathscr{F}(t) = -\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|},$$

and

$$\frac{A(u_A, v_C, v_D)}{V} \leqslant C\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|}.$$

Subcase (b). The second subcase corresponds the situation described by Fig. 7. In this case, before α_1 crosses α_2 at time t, we have $\sigma(\alpha_1) > \sigma(u_B, v_C)$, $\sigma(\alpha_2) < \sigma(u_B, v_D)$. And right after the crossing at time t, we have $\sigma(\alpha_1) > \sigma(u_B, v_D)$, $\sigma(\alpha_2) < \sigma(u_A, v_D)$. Therefore, by the definition of the nonlinear functional, we have

$$L(u,v)\big(x_{\alpha_1}, t^+\big) = L(u,v)\big(x_{\alpha_1}, t^-\big) \neq L(v,u)\big(x_{\alpha_2}, t^+\big) = L(v,u)\big(x_{\alpha_2}, t^-\big).$$

Hence,

$$\begin{aligned}
&E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\
&\quad = \bigg[\bigg(\frac{|\alpha_1||u_B - v_D||\sigma(\alpha_1) - \sigma(u_B, v_D)| - |\alpha_1||u_B - v_C||\sigma(\alpha_1) - \sigma(u_B, v_C)|}{V}\bigg)L(u,v)\big(x_{\alpha_1}, t^+\big)\\
&\qquad + \bigg(\frac{|\alpha_2||u_A - v_C||\sigma(\alpha_1) - \sigma(u_A, v_C)| - |\alpha_2||u_B - v_C||\sigma(\alpha_2) - \sigma(u_B, v_C)|}{V}\bigg)L(v,u)\big(x_{\alpha_2}, t^+\big)\bigg]\\
&\qquad + K\Delta\mathscr{F}(t)\big\|u(.,t) - v(.,t)\big\|_{L^1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&|\alpha_1||u_B - v_D|\big|\sigma(\alpha_1) - \sigma(u_B, v_D)\big| - |\alpha_1||u_B - v_C|\big|\sigma(\alpha_1) - \sigma(u_B, v_C)\big|\\
&\quad = |\alpha_1||u_B - v_C|\big|\sigma(\alpha_1) - \sigma(u_B, v_C)\big| + |\alpha_1||\alpha_2|\big(\sigma(\alpha_1) - \sigma(\alpha_2)\big) - |\alpha_1||u_B - v_C|\big|\sigma(\alpha_1) - \sigma(u_B, v_C)\big|\\
&\quad = |\alpha_1||\alpha_2|\big(\sigma(\alpha_1) - \sigma(\alpha_2)\big).
\end{aligned}$$

Similarly, we have

$$|\alpha_2||u_A - v_C|\big|\sigma(\alpha_1) - \sigma(u_A, v_C)\big| - |\alpha_2||u_B - v_C|\big|\sigma(\alpha_2) - \sigma(u_B, v_C)\big| = |\alpha_1||\alpha_2|\big(\sigma(\alpha_1) - \sigma(\alpha_2)\big).$$

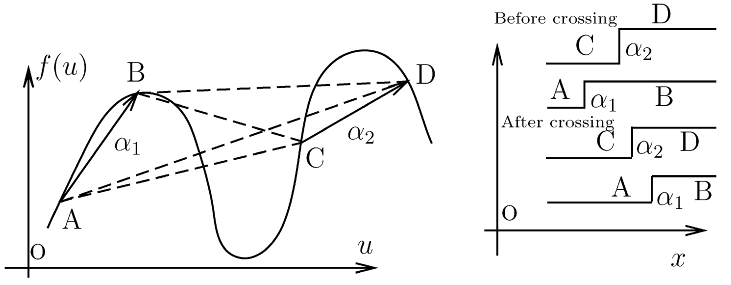

Fig. 8. Subcase (c) in Case 3 of Theorem 2.4.

Since

$$\Delta\mathscr{F}(t) = -\frac{|\alpha_1||\alpha_2|(\sigma(\alpha_1)-\sigma(\alpha_2))}{|\alpha_1|+|\alpha_2|},$$

we have

$$E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big) \leqslant 0.$$

Subcase (c). As shown in Fig. 8, before α_1 crosses α_2 at time t, we have $\sigma(\alpha_1) > \sigma(u_B, v_C)$, $\sigma(\alpha_2) > \sigma(u_B, v_D)$. And right after α_1 crosses α_2, we have $\sigma(\alpha_1) > \sigma(u_B, v_D)$, $\sigma(\alpha_2) > \sigma(u_A, v_D)$. It is straightforward to check that in this case,

$$L(u,v)\big(x_{\alpha_1}, t^+\big) = L(u,v)\big(x_{\alpha_1}, t^-\big) = L(v,u)\big(x_{\alpha_2}, t^+\big) = L(v,u)\big(x_{\alpha_2}, t^-\big).$$

Thus,

$$\begin{aligned}\Delta E(u,v)(t) &= E(u,v)\big(t^+\big) - E(u,v)\big(t^-\big)\\ &\leqslant \frac{[B(u_A,u_B,v_D)+B(u_A,v_C,v_D)-B(u_A,u_B,v_C)-B(u_B,v_C,v_D)]L(u,v)(x_{\alpha_1},t^+)}{V}\\ &\quad + K\Delta\mathscr{F}(t)\big\|u(.,t)-v(.,t)\big\|_{L^1} \leqslant 0,\end{aligned}$$

where the constant K is chosen to be suitably large. In summary, this completes the proof of Theorem 2.4. $\square$

Remark 4.1. In fact, one can check that there are other subcases for Case 3 in the proof for Theorem 2.4. For example, if we assume α_1 is on the left of α_2 with $\sigma(\alpha_1) > \sigma(\alpha_2)$ and there is no intersection between these two waves on the u–$f(u)$ plane, there are four possibilities in Case 3, that is, the cases when $u_A < u_B < v_C < v_D$, $u_A < u_B < v_D < v_C$, $u_B < u_A < v_C < v_D$ and $u_B < u_A < v_D < v_C$ respectively. In each of these cases, according to the definition of $L(u,v)(x_{\alpha_1}, t^-)$, $L(u,v)(x_{\alpha_1}, t^+)$, $L(u,v)(x_{\alpha_2}, t^-)$, $L(u,v)(x_{\alpha_2}, t^+)$, there are three subcases depending on the relative velocities of α_1 and α_2, cf. Fig. 9. On the other hand, if α_1 and α_2 have intersection on the u–$f(u)$ plane, by the entropy condition there are less possibilities. In summary, one can check all the cases by using the argument in Theorem 2.4 and show that $E(u,v)(t)$ is non-increasing through the wave interactions and wave crossings.

Finally, we give the proof of Theorem 2.5 as follows.

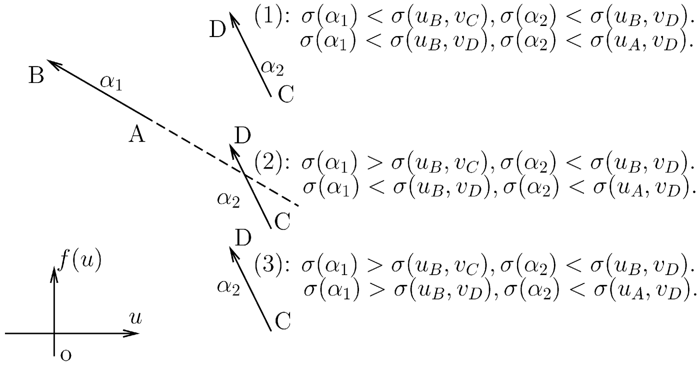

Fig. 9. Remark 4.1.

Proof of Theorem 2.5. Take any wave $\alpha \in \Phi(u)$ and define

$$E_{\alpha_k}(t) = B\big(u(x_{\alpha_k}-,t), u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)L(u,v)(x_{\alpha_k},t) + K\mathscr{F}_{\alpha_k}(t)\big\|u(.,t) - v(.,t)\big\|_{L^1},$$

where $\mathscr{F}_{\alpha_k}(t)$ is part of $\mathscr{F}(t)$ related to the wave α_k which is one of the partitioned waves in α, that is,

$$\mathscr{F}_{\alpha_k}(t) = \Sigma_\beta \frac{\sum_l |\alpha_k||\beta_l||\sigma(\alpha_k) - \sigma(\beta_l)|\chi((x_{\alpha_k} - x_{\beta_l})(\sigma(\beta_l) - \sigma(\alpha_k)))}{T.V.(\alpha,\beta)} + K_1|\alpha_k|V(t),$$

where β is any wave either in the solution u or v.

Notice that $E_{\alpha_k}(t)$ is differentiable and $\mathscr{F}_{\alpha_k}(t)$ is constant in the time interval when there is no wave interaction or wave crossing.

In such a time interval, firstly, we have

$$\frac{\partial L(u,v)(x_{\alpha_k},t)}{\partial t} = -\big|u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)\big|\big|\sigma(\alpha_k) - \sigma\big(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)\big|,$$

and

$$\frac{\partial L(v,u)(x_{\alpha_k},t)}{\partial t} = -\big|u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)\big|\big|\sigma(\alpha_k) - \sigma\big(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)\big|.$$

Thus,

$$\frac{\partial E_{\alpha_k}}{\partial t} = -\frac{|\alpha_k||u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)|^2|\sigma(\alpha_k) - \sigma(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t))|^2}{V(t)}.$$

When wave interaction or wave crossing happens, from the estimation on the functional $E(u,v)(t)$, we know that it is non-increasing. On the other hand, in the wave front tracking scheme, for any time T, there is only finitely many wave interactions and wave crossings. Thus, we can conclude that

$$\begin{aligned}
&\int_0^t \frac{|\alpha_k||u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)|^2|\sigma(\alpha_k) - \sigma(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t))|^2}{V(t)}\,dt \\
&\quad\leqslant CE_{\alpha_k}(u,v)(0) \\
&\quad\leqslant C|\alpha_k|V(0)\|u_0 - v_0\|_{L^1}.
\end{aligned}$$

Thus, by Cauchy–Schwarz inequality, we have

$$\int_0^t |\alpha_k| \big|u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)\big| \big|\sigma(\alpha_k) - \sigma\big(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)\big|\,dt$$

$$\leqslant E_{\alpha_k}(u,v)(0)^{\frac{1}{2}} \left[\int_0^t |\alpha_k| V(t)\,dt\right]^{\frac{1}{2}}$$

$$\leqslant C|\alpha_k| V(0)^{\frac{1}{2}} \|u_0 - v_0\|_{L^1}^{\frac{1}{2}} \left[\int_0^t V(t)\,dt\right]^{\frac{1}{2}}.$$

Finally, we obtain

$$\sum_{\alpha\in\Phi(u)\cup\Phi(v)} \sum_k \int_0^t |\alpha_k| \big|u(x_{\alpha_k}+,t) - v(x_{\alpha_k}+,t)\big| \big|\sigma(\alpha_k) - \sigma\big(u(x_{\alpha_k}+,t), v(x_{\alpha_k}+,t)\big)\big|\,dt$$

$$\leqslant C V(0)^{\frac{3}{2}} \|u_0 - v_0\|_{L^1}^{\frac{1}{2}} \left[\int_0^t V(t)\,dt\right]^{\frac{1}{2}},$$

and this completes the proof of Theorem 2.5. □

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